

Note on Matrix Model with Massless Flavors

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ABSTRACT: In this note, following the work of Seiberg in hep-th/0211234 for the conjecture between the field theory and matrix model in the case with massive fundamental flavors, we generalize it to the case with massless fundamental flavors. We show that with a little modifications, the analysis given by Seiberg can be used directly to the case of massless flavors. Furthermore, this new method explains the insertion of delta functions in the matrix model given by Demasure and Janik in hep-th/0211082.

KEYWORDS: Matrix Model, Massless Flavors.

Contents

1. Introduction

The field theory *v.s* matrix model conjecture proposed by Dijkgraaf and Vafa [1, 2, 3] has caught a lot of attentions recently (from [4] to [18]). The beautiful point of this conjecture is that a lot of non-trivial complex calculations of exact low energy superpotential in the field theory side can be easily done in the corresponding matrix model. Even in the case that there is no exact solution in the matrix model, perturbative calculation is still much easier than the one in the corresponding field theory. For example, the Montonen-Oliver duality can be derived (at least perturbatively) by matrix model in [4]¹.

Although the matrix model conjecture is inspired by studies of geometric engineering of field theories in string theory, it can be proved in the pure field theory. Two recent papers [5, 6] laid solid grounds for this conjecture. It was realized that properties of chiral rings and anomalies are very powerful tools to uniquely determine the exact low energy effective superpotential. These two papers dealt with the prototype example, i.e., $\mathcal{N} = 1$ $U(N)$ gauge theory with one adjoint field Φ and tree level superpotential deformed from the corresponding $\mathcal{N} = 2$ theory. Using same idea in [6] Seiberg has generalized the field theory proof of matrix model conjecture to the massive fundamental flavors [7]. The result explained why field theory calculations are recaptured by matrix models studies in [9, 10, 11, 16].

In [7], it was assumed that the matrix $m(z)$ is not degenerated, so all quarks are massive and are integrated out. Therefore, the low energy effective action does not depend on the meson operator. The result is related to the fact that polynomials $q(z)$ has been fixed when solving the loop equation. It was also remarked in [7] that an ambiguity in $q(z)$ will sign the appearance of more fields in the effective superpotential.

In this note, we will try to do a little more than that in [7] by letting matrix $m(z)$ to be degenerated. Under the new situation, the low energy effective action will be a function of not only glueball fields S_i , but also some massless fields (mesons M_i^j). In fact, we will show that with just a little bit of modifications, the method of [7] can be applied to our new cases.

The organization of the note is following. In section two we briefly review the result in [7]. In section three we make our proposal to the study of massless flavors. In section four we give conclusion and discussion.

¹We want to thank Freddy Cachazo for discussing about this point.

2. Review of results in [7]

In [7], Seiberg considered a general $\mathcal{N} = 1$ $U(N)$ theory with the adjoint field Φ , N_f fundamental Q^i and N_f anti-fundamental \tilde{Q}_j . The tree level superpotential is

$$W_{tree} = \text{Tr } W(\Phi) + \tilde{Q}_i m_j^i(\Phi) Q^j \quad (2.1)$$

with the function $W(\Phi)$ and matrix $m_j^i(\Phi)$ to be following polynomials

$$W(z) = \sum_{k=1}^n \frac{g_k}{k+1} z^{k+1} \quad (2.2)$$

$$m_j^i(z) = \sum_{k=1}^{l+1} m_{j,k}^i z^{k-1} \quad (2.3)$$

To solve the theory, it is useful to define following chiral operators

$$T(z) = \text{Tr} \left(\frac{1}{z - \Phi} \right), \quad (2.4)$$

$$w_\alpha(z) = \frac{1}{4\pi} \text{Tr} \left(\frac{W_\alpha}{z - \Phi} \right), \quad (2.5)$$

$$R(z) = \frac{-1}{32\pi^2} \text{Tr} \left(\frac{W_\alpha W^\alpha}{z - \Phi} \right), \quad (2.6)$$

$$M_i^j = \tilde{Q}_i \frac{1}{z - \Phi} Q^j \quad (2.7)$$

which will satisfy following loop equations given by anomaly considerations

$$[W'(z)T(z)]_- + \text{tr}[m'(z)M(z)]_- = 2R(z)T(z) + w_\alpha(z)w^\alpha(z) \quad (2.8)$$

$$[W'(z)w_\alpha(z)]_- = 2R(z)w_\alpha(z) \quad (2.9)$$

$$[W'(z)R(z)]_- = R(z)^2 \quad (2.10)$$

$$[(M(z)m(z))_i^j]_- = R(z)\delta_i^j \quad (2.11)$$

$$[(m(z)M(z))_i^j]_- = R(z)\delta_i^j \quad (2.12)$$

where tr is over the flavor index while Tr , over the color index. Among these equations (2.10) is same as in the case of pure adjoint field and can be solved by

$$W'(z)R(z) + \frac{1}{4}f(z) = R(z)^2 \quad (2.13)$$

with $f(z) = -4[W'(z)R(z)]_+$ of degree $(n-1)$. Equations (2.11) and (2.12) can be written as

$$(M(z)m(z) - q(z))_i^j = R(z)\delta_i^j, \quad q(z) = [M(z)m(z)]_+ \quad (2.14)$$

$$(m(z)M(z) - \tilde{q}(z))_i^j = R(z)\delta_i^j, \quad \tilde{q}(z) = [m(z)M(z)]_+ \quad (2.15)$$

with consistent condition $\tilde{q} = m q m^{-1}$. The solution of (2.14) (2.15) can be written as

$$M(z) = R(z)m^{-1}(z) + q(z)m^{-1}(z) \quad (2.16)$$

To avoid singularities in the solution (2.16) when $m(z)$ has a zero eigenvalue, $q(z)$ has been uniquely fixed to remove these singularities.

If we write $W'(z) = g_n \prod_{i=1}^n (z - a_i)$, in the case that $m(a_i) \neq 0$ for all a_i , all quarks are massive and should be integrated out. The low energy field theory is just the pure SUSY Yang-Mills theory and the remained massless fields are these S_i, ω_α^i . Using the similar counting as in [5, 6], it can be shown that only genus two and genus one can contribute. The answer for the effective superpotential is given by

$$W_{eff}(S_i, w_\alpha^i, N_i) = \int d^2\psi \mathcal{F}_0(S_i(\psi)) + \mathcal{F}_1(S_i) \quad (2.17)$$

with

$$\mathcal{F}_0(S_i(\psi)) = \mathcal{F}_0^{pert} + \frac{1}{2} \sum_i S_i^2 (\log \frac{\Lambda^3}{S_i} + \frac{3}{2}) \quad (2.18)$$

$$\mathcal{F}_1(S_i) = \mathcal{F}_1^{pert}(S_i) \quad (2.19)$$

to be contributions of sphere and disk respectively.

The corresponding matrix model for above field theory (2.1) is given by free energy $\hat{\mathcal{F}}$ through

$$\exp(-\frac{\hat{N}^2}{\hat{g}^2} \hat{\mathcal{F}}) = \int d\hat{\Phi} d\hat{Q} d\hat{\bar{Q}} \exp(-A) \quad (2.20)$$

with the action

$$A = \frac{\hat{N}}{\hat{g}} [\text{Tr } W(\hat{\Phi}) + \hat{\bar{Q}}_i m_j^i(\hat{\Phi}) \hat{Q}^j] . \quad (2.21)$$

where $\hat{\Phi}$ is $\hat{N} \times \hat{N}$ matrix, $\hat{\bar{Q}}$, $N_f \times \hat{N}$ matrix and \hat{Q} , $\hat{N} \times N_f$ matrix. The relevant resolvents are given by

$$\hat{R}(z) = \frac{\hat{g}}{\hat{N}} \langle \text{Tr}(\frac{1}{z - \hat{\Phi}}) \rangle \quad (2.22)$$

$$\hat{M}_i^j(z) = \langle \hat{\bar{Q}}_i \frac{1}{z - \hat{\Phi}} \hat{Q}^j \rangle \quad (2.23)$$

In the large \hat{N} limit with fixed N_f , loop equations in the matrix model become

$$[W'(z) \hat{R}(z)]_- = \hat{R}(z)^2 \quad (2.24)$$

$$[(\hat{M}(z) m(z))_i^j]_- = \hat{R}(z) \delta_i^j \quad (2.25)$$

$$[(m(z) \hat{M}(z))_i^j]_- = \hat{R}(z) \delta_i^j \quad (2.26)$$

which exactly correspond to these in the field theory (2.10) (2.11) (2.12). From these we can identify

$$\hat{R}(z) = \langle R(z) \rangle, \quad \hat{M}(z) = \langle M(z) \rangle \quad (2.27)$$

We need to separate out the sphere and disk contributions to free energy in the matrix model by $\hat{\mathcal{F}}_0 = \lim_{\hat{N} \rightarrow \infty} \hat{\mathcal{F}}$, $\hat{\mathcal{F}}_1 = \lim_{\hat{N} \rightarrow \infty} \frac{\hat{N}}{\hat{g}} (\hat{\mathcal{F}} - \hat{\mathcal{F}}_0)$. Then it can be shown that

$$\mathcal{F}_0 = \hat{\mathcal{F}}_0, \quad \mathcal{F}_1 = \hat{\mathcal{F}}_1 . \quad (2.28)$$

Thus it established the equivalence between the gauge theory and the matrix model.

3. The degenerated matrix $m_i^j(z)$

As mentioned in the introduction, the derivation in the section two is under the assumption that $m(\Phi)$ is a generic non-degenerated matrix. If the matrix $m(\Phi)$ is degenerated, some flavors will be massless and the low energy effective action will involve more variables than these S_i, w_α^i . However, since the $SU(N_i)$ parts are confined and there are only $U(1)$ parts left, the proper low energy variables are not Q^i, \tilde{Q}_j , but meson fields $M_j^i = \tilde{Q}_j Q^i$ which are singlets of $SU(N_i)$.

To get the effective superpotential as a function of S_i, w_α^i and M_j^i , we will adopt a trick used in [8]. In that paper, to study the field theory of double trace deformation

$$W(\Phi) = \frac{g_2}{2} \text{Tr}(\Phi^2) + g_4 \text{Tr}(\Phi^4) + \tilde{g}_2 (\text{Tr}(\Phi^2))^2 \quad (3.1)$$

and the corresponding matrix model, we linearized it by involving an additional gauge singlet field A with superpotential

$$W_{tree} = \frac{1}{2}(g_2 + 4\tilde{g}_2 A) \text{Tr}(\Phi^2) + g_4 \text{Tr}(\Phi^4) - \tilde{g}_2 A^2 \equiv W_{single} - \tilde{g}_2 A^2 \quad (3.2)$$

Now, except the last term $-\tilde{g}_2 A^2$ which does not include the field Φ , W_{single} is in the typical form of cases discussed in [5, 6]. According to the prescription in these two papers, what we need to do is just to integrate field Φ while *leaving field A untouched*². In another word, the exact effective action is given by minimizing

$$W_{single}^{(exact)}(A, S_i) - \tilde{g}_2 A^2 \quad (3.3)$$

regarding to A . The $W_{single}^{(exact)}$ can be calculated by field theory or matrix model methods. If we use the matrix model method, we need only to use W_{single} in (3.2) to calculate the free energy \mathcal{F}_0 and get

$$W_{single}^{(exact)}(A, S_i) = N_i \frac{\partial \mathcal{F}_0(S_i, A)}{\partial S_i} \quad (3.4)$$

where measure term has been included into $\mathcal{F}_0(S_i, A)$ implicitly.

Same logic tells us how to deal with the degenerated matrix m . What we need to do is to involve some new fields X_i^j such that $\tilde{m}(\Phi, X)$ is not degenerated. Therefore the analysis in the section two can be applied and corresponding matrix model can be derived. More explicitly, the tree level superpotential will be the form

$$\tilde{W} = W(\Phi, Q, \tilde{Q}, X) + L(X) \quad (3.5)$$

where matrix $\tilde{m}(\Phi, X)$ in $W(\Phi, Q, \tilde{Q}, X)$ will be non-degenerated. Now to the part $W(\Phi, Q, \tilde{Q}, X)$, X can be treated as parameters and situation is reduced to the standard one in (2.1). It is not hard to see that every step in [7] is valid and need no modifications:

²We like to thank Freddy Cachazo for emphasizing this point to us.

we just integrate all Φ , Q and \tilde{Q} (but do not touch X). Especially the action for the corresponding matrix model is just given by $W(\Phi, Q, \tilde{Q}, X)$. From this matrix model we can calculate $\hat{\mathcal{F}}_k(S_i, w_\alpha^i, X)$, $k = 0, 1$. The only modification of above procedures is that we need to minimize the action

$$W^{(exact)}(\Phi, Q, \tilde{Q}, X) + L(X) \quad (3.6)$$

regarding to X . It is exactly this minimization giving the final effective action as a function of S_i, w_α^i as well as the meson fields.

To demonstrate our idea, let us use one simple example with only N_f flavors and degenerated mass matrix $m = \begin{bmatrix} m_{K \times K} & 0 \\ 0 & 0 \end{bmatrix}$. This simple model has been discussed in [9, 10], but now we use our new understanding to redo it. Involving the new field X_i^j , the superpotential will become

$$W = \sum_{i,j=1}^K \tilde{Q}_i m_j^i Q^j + \sum_{a,b=1}^{N_f-K} X_a^b (-M_b^a + \tilde{Q}_b Q^a) \equiv \sum_{i,j=1}^{N_f} \tilde{Q}_i \tilde{m}_j^i Q^j - \text{tr}(XM) \quad (3.7)$$

where X can be treated as Lagrangian multiplier, M meson fields and $\tilde{m} = \begin{bmatrix} m_{K \times K} & 0 \\ 0 & X_a^b \end{bmatrix}$. With the new non-degenerated mass matrix \tilde{m} , we can apply the analysis in [7] and reduce the field theory problem to the matrix model. Using (2.20) and (2.21) the matrix integration is

$$\exp(-\frac{\hat{N}^2}{\hat{g}^2} \hat{\mathcal{F}}) = \int d\hat{Q} d\tilde{Q} \exp(-\frac{\hat{N}}{\hat{g}} \tilde{Q}_i \tilde{m}_j^i \hat{Q}^j) \quad (3.8)$$

This matrix integration has been done in [9, 10] and we got

$$W_{exact} = N_c (\det(\tilde{m}) \Lambda^{3N_c - N_f})^{\frac{1}{N_c}} - \text{tr}(XM) = N_c (\det(m) \det(X) \Lambda^{3N_c - N_f})^{\frac{1}{N_c}} - \text{tr}(XM) \quad (3.9)$$

where the first term is got from the matrix model. Next thing we need to do is to minimize superpotential (3.9) regarding to X . Differentiating $\frac{\partial}{\partial X}$ we get

$$M = X^{-1} (\det(m) \det(X) \Lambda^{3N_c - N_f})^{\frac{1}{N_c}} \quad (3.10)$$

Taking determinant of (3.10) at two sides, we get

$$\det(X) = \frac{(\det(M))^{\frac{N_c}{N_f - K - N_c}}}{(\det(m) \Lambda^{3N_c - N_f})^{\frac{N_f - K}{N_f - K - N_c}}} \quad (3.11)$$

Putting (3.11) back into (3.10) we solve

$$X = M^{-1} \left(\frac{\det(M)}{\det(m) \Lambda^{3N_c - N_f}} \right)^{\frac{1}{N_f - K - N_c}} \quad (3.12)$$

So finally we have

$$W_{exact} = (N_c - (N_f - K)) \left(\frac{\det(m) \Lambda^{3N_c - N_f}}{\det(M)} \right)^{\frac{1}{N_c - (N_f - K)}} \quad (3.13)$$

This is exactly the result got in [10] (notice that K in this paper should be $N_f - K$ in equation (2.5) of [10]). From above calculations, we see how the dependence of W on meson fields can be recovered from the minimization of field X .

We can repeat same calculation before the glueball field S has been integrated out. The effective action is given in [9] as

$$W = -N_c [S \log \left(\frac{S}{(\det(m) \det(X) \Lambda^{3N_c - N_f})^{\frac{1}{N_c}}} \right) - S] - \text{tr}(XM) \quad (3.14)$$

Minimizing it regarding to X , we get

$$X = SM^{-1} \quad (3.15)$$

Putting it back into (3.14) we find

$$W = -(N_c - (N_f - K)) [S \log \left(\frac{S}{\left(\frac{\det(m) \Lambda^{3N_c - N_f}}{\det(M)} \right)^{\frac{1}{N_c - (N_f - K)}}} \right) - S] \quad (3.16)$$

which is same form found in [10] (equation (3.5)).

There are several points we like to remark. First, results in [9, 10] are got by the insertion of proper delta-functions while here, by involving new fields X . In fact, this new method shed light on the origin of delta functions. In our new method, we minimize X after the integration in matrix model. However, in our example the contribution of X to the exact effective action comes from \mathcal{F}_1 term only, i.e., $W = \mathcal{F}_1 + L(x) + \dots$, where ... are terms do not have X . Minimizing W relative to X is equal to minimizing $\mathcal{F}_1 + L(x)$ in matrix model, which in turn is equal to integrating X fields in the matrix model using the full action (including the $\text{tr}(XM)$ terms) directly. Applying the $\int dX$ to action (3.7) we get immediately the delta function

$$\delta(M_b^a - \tilde{Q}_b Q^a) . \quad (3.17)$$

So the analysis given by Seiberg explains the proposal made in [10]. The delta function (3.17) can be understood from another point of view: it changes the coordinate Q, \tilde{Q} in UV theory to proper coordinate M in IR^3 . We want to emphasize that in general situations (for example the baryonic deformation in superpotential), minimizing X at the level of effective action is not same as integrating X at the level of free energy in matrix model (for example, the double trace deformation in [8]), so the insertion of delta functions will be modified.

³This point of view is also given by K. Ohta in [14] where the appearance of delta function is explained from this point of view.

Second, our example (3.7) does not apply to the case $N_c < N_f - K$ because in this case, the number of parameters M_a^b is more than the one of physical parameters in the moduli space⁴. To deal with these extra parameters, we can involve extra fields Y as multipliers of constraints and minimize the effective action relative to Y . Another way is to treat X_a^b not totally independent to each other when minimizing the action. No matter which method to be used, it is more involving, but also promising to solve the matrix model with baryonic deformation.

Third, when $N_c = N_f - K$, (3.14) and (3.16) reduce to

$$W = S \log\left(\frac{\det(m)\Lambda^{3N_c-N_f}}{\det(M)}\right) \quad (3.18)$$

which, after minimizing S , gives $W = 0$ and

$$\det(m)\Lambda^{3N_c-N_f} = \det(M) \quad (3.19)$$

as the quantum corrected moduli space [11, 13].

4. Conclusions

In this paper we made a little step along the analysis given by Seiberg in [7] for the field theory *v.s.* matrix model conjecture in the case with fundamental flavors. We allow the matrix $m(\Phi)$ to be degenerated so that the low energy effective superpotential will depend on both the glueball fields S_i, w_α^i and other massless fields, for example, mesons M . We showed that, by involving new fields X as Langrangian multipliers, the analysis in [7] can be translated into this new case with just a little modification. Furthermore this new method explains the insertion of delta function in the matrix model integrations proposed in [11, 10].

It is obvious that there are still a lot of works needed to be done. For example, it will be nice to generalize the discussion to the superpotential with baryonic like deformations. The case of multi gauge groups is also interesting because it will shed light on various things, like chiral fields and duality. One of most important things is still the investigation of dualities in the matrix model. It is very interesting to see if the matrix model can fascinate the study of dualities in field theories.

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⁴For case $N_c < N_f - K$, parameters M are $(N_f - K)^2$ while by Higgsing mechanism, there are only $2(N_f - K)N_c - N_c^2$ massless fields. For case $N_c \geq N_f - K$, by Higgsing mechanism, there are $2(N_f - K)N_c - (N_c^2 - (N_c - N_f + K)^2) = (N_f - K)^2$, so it is equal to the number of meson fields M .

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